FORTY-THIRD ANNUAL
AUGUSTA UNIVERSITY
MATHEMATICS CONTEST
MARCH 3, 2017
WRITTEN EXAM

INSTRUCTIONS

(To be read aloud by proctor)

Use a #2 pencil only. Do not use a pen or a colored pencil. Turn your answer sheet so that the word "NAME" is at the top. In the spaces provided, print your last name, first name, and middle initial. Enter today's date, which is March 15, 2013. Print the name of your school in the space marked "SUBJECT", and in the same space indicate whether you are on Team A, Team B, or an alternate.

Turn your answer sheet so that the numbers are in the proper order.

This is a multiple-choice test. You are to mark only one answer to each question. Answers must be recorded on the separate answer sheet. Select the one best answer to each question and record this answer by blackening the corresponding space to the appropriate letter on the answer sheet. For example, if you select answer b for the first question, then you would blacken the "b" on the answer sheet for that first question. There is NO penalty for guessing. Do not make any stray marks or scratch work on the answer sheet.

You may do any scratch work on the space provided on these pages.

Do as many problems as you can in the 60 minutes allowed. **You may not leave the room until 60 minutes is over.** When the alarm sounds on the timer or when the proctor requests you to stop, please cease work immediately, put your pencil down, and turn your answer sheet over.

Essentially all of the problems require some calculations. Do not be hasty in your judgments, but work as rapidly as possible. For each problem you should work out ideas on paper before selecting the answer.

There are 20 questions on the test. Check your test carefully before starting. If a question or page is missing, raise your hand and the proctor will provide you with a correct copy of the test. In case of ties, the answers to Number 20 will be examined to break the tie. If there is still a tie, the answers to Number 19 will be examined, etc.

The proctor is NOT permitted to explain to you the meaning of any question, so do not request your proctor to break the rules of the contest. **The use of calculators is prohibited.** If you have questions concerning the instructions, ask them now.

DO NOT TURN THE PAGE UNTIL THE PROCTOR TELLS YOU TO BEGIN WORK.

1. Suppose the following statements are accepted as true:

   (I) No birds, except ostriches, are 9 feet high.
   (II) There are no birds in this aviary that belong to any one but me.
   (III) No ostrich lives on mince-pies.
   (IV) I have no birds less than 9 feet high.

   Which of the following statements is a valid conclusion?

   (a) All birds belonging to me live on mince-pies.
   (b) The only birds in this aviary are 9 feet high.
   (c) Birds living on mince-pies are always 9 feet high.
   (d) No bird in this aviary lives on mince-pies.
   (e) None of the above.

2. Find all solutions of the following equation lying in the interval $[0, 2\pi)$:

$$(2\sin x - 1)(\sin x + 1) = \cos x$$

   (a) $\{0, \pi/2, 3\pi/4, 5\pi/4\}$
   (b) $\{\pi/4, \pi, 5\pi/4, 3\pi/2\}$
   (c) $\{\pi/4, \pi, 3\pi/4, 7\pi/4\}$
   (d) The equation has no solutions.
   (e) None of the above.

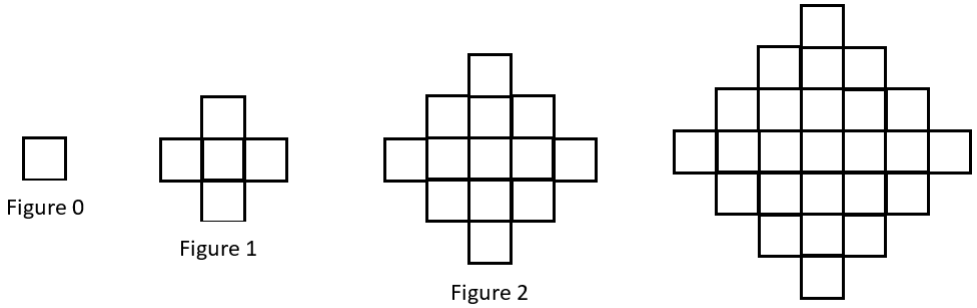
3. The sum of the lengths of the three sides of a right triangle is 56. The sum of the squares of the lengths of the three sides of the same right triangle is 1250. What is the area of the triangle?

   (a) 76
   (b) 80
   (c) 84
   (d) 92
   (e) None of the above.

4. Suppose that $3 = k2^r$ and that $15 = k4^r$. Then r equals:

   (a) $-\log_2 5$
   (b) $\log_5 2$
   (c) $\log_{10} 5$
   (d) $\log_2 5$
   (e) $5/2$

5. Figures 0, 1, 2, and 3 consist of 1, 5, 13, and 25 non-overlapping unit squares, respectively. If the pattern were continued, how many non-overlapping unit squares would there be in Figure 100?



(a) 10,401
(b) 19,801
(c) 20,201
(d) 39,801
(e) 40,801

6. For all real numbers x, except $x = 0$ and $x = 1$, the function f is defined by

$$f\left(\frac{x}{x-1}\right) = \frac{1}{x}$$

Suppose $0 < \theta < \dfrac{\pi}{2}$. What is $f\left(\sec^2\theta\right)$?

(a) $\sin^2\theta$
(b) $\cos^2\theta$
(c) $\tan^2\theta$
(d) $\cot^2\theta$
(e) $\csc^2\theta$

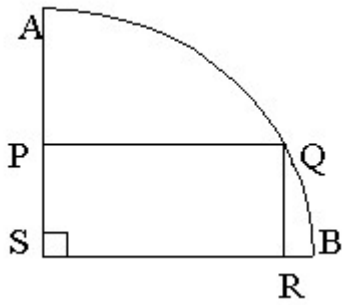
7. Determine the number of integers not exceeding 50 that are divisible by 2 or by 3 or by 5.

   (a) 37
   (b) 36
   (c) 35
   (d) 39
   (e) 38

8. Compute $1000^{\log 2+\log 3}$, where the logarithms are base 10.

   (a) 18
   (b) 125
   (c) 216
   (d) 5000
   (e) 6000

9. ASB is a quarter circle. PQRS is a rectangle with sides PQ = 8 and PS = 6. What is the length of the arc AQB ?



   (a) 5π
   (b) 10π
   (c) 25
   (d) 14
   (e) 28

10. If $f(x+1) = x^2 - 1$, then $f(x)$ equals:

    (a) $x^2 - 2x + 1$
    (b) $x^2 + 2x + 1$
    (c) $x^2 - 2x$
    (d) $x^2 + 2x$
    (e) x^2

11. The value of $\dfrac{13! - 12!}{11!}$ is:

    (a) 121
    (b) 156
    (c) 144
    (d) 132
    (e) 169

12. Among the positive integers less than 100, each of whose digits is a prime number, one is selected at random. What is the probability that the selected number is prime?

    (a) 8/99
    (b) 2/5
    (c) 9/20
    (d) 1/2
    (e) 9/16

13. The y-intercepts, P and Q, of two perpendicular lines intersecting at the point $A(6, 8)$ have a sum of zero. What is the area of ΔAPQ?

    (a) 45
    (b) 48
    (c) 54
    (d) 60
    (e) 72

14. Given $xy = x + y$, $yz = y + z$ and $xz = x + z$, the value of $\dfrac{xyz}{xy + xz + yz}$ equals:

    (a) 2/3
    (b) 3/2
    (c) 1/2
    (d) 2
    (e) 1

15. A towing company stores cars on a large square lot. The company is constructing an office building in one corner of this lot, covering an area measuring 50 ft $\times$ 50 ft . If 13,125 sq.ft. will remain available for storing cars, what are the dimensions of the company's large lot?

    (a) 125 ft $\times$ 125 ft
    (b) 120 ft $\times$ 120 ft
    (c) 130 ft $\times$ 130 ft
    (d) 115 ft $\times$ 115 ft
    (e) 135 ft $\times$ 135 ft

16. There are 50 balls in a box, all of which are blue or red. If the probability of drawing two red balls in a row (with replacement) is 256/625 , how many of the balls are red?

    (a) 30
    (b) 31
    (c) 32
    (d) 33
    (e) 34

17. A **derangement** of the digits $1, 2, 3, \ldots, n$ is a permutation of those n digits where 1 does not appear as the first digit (or in other words, 1 is not in its natural position), 2 does not appear as the second digit, and in general none of the digits appear in their natural position. Note that 21 is the only derangement of 2 digits, 312 and 231 are the only derangements of three digits, and you can check there are nine derangements of four digits. Thus, find the number of derangements of five digits.

    (a) 43
    (b) 44
    (c) 45
    (d) 46
    (e) 47

18. Find the remainder when 2^{2017} is divided by 5.

    (a) 0
    (b) 1
    (c) 2
    (d) 3
    (e) 4

19. Find the probability that a randomly picked integer x from the set $\{1, 2, \ldots, 7^{10000}\}$ will satisfy $gcd\,(x, 7^{10000}) = 1$

    (a) $\dfrac{1}{7}$
    (b) $\dfrac{7^{10000} - 1}{7^{10000}}$
    (c) $\dfrac{7^{10000} - 10000}{7^{10,000}}$
    (d) $\dfrac{6}{7}$
    (e) None of the above.

20. The equation $x^4 + 7x^3 + 10 = 0$ has 4 solutions (counting multiplicities) in the complex numbers. This is the number of distinct integer solutions to that equation.

    (a) 0
    (b) 1
    (c) 2
    (d) 3
    (e) 4